Mathematical foundations of Infinite-Dim Statistical models

Chap.2.6.2∼2.6.3

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2018.08.31

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Reviwe of RKHS

For centred Gaussian Process $X_t, t \in T$, Define $F := span\{X_t : t \in T\} \subset L^2(\Omega, \Sigma, \mathbb{P})$

Definition 2.6.1(RKHS of GP) The reproducing kernel Hilbert space of a centered Gaussian process is

$$H = completion(\{\mathbb{E}(hX) : h \in F\})$$

Definition 2.6.4(RKHS of B-valued random variable) Let B be a separable Banach space, and let X be a B-valued centered random variable. Define $F = \{f(X) : f \in B^*\} \subset L^2(\Omega, \Sigma, \mathbb{P})$ and $\bar{F}$ is its completion. The reproducing kernel hilbert space of X is

$$H = \{\mathbb{E}hX : h \in \bar{F}\} \subset B$$

with inner product $\langle \mathbb{E}(h_1 X), \mathbb{E}(h_2 X) \rangle_H := \mathbb{E}h_1 h_2$

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- Isoperimetric inequality
- Equivalence and Singularity
- Small ball estimation

Isoperimetric

Theorem 2.6.12 Let O_H be the unit ball centerd at zero of the RKHS H of X, where X is a centered Gaussian B-valued random variable, B a separable Banach space. Let μ be the probability law of X. Then, for every set $A \in \mathcal{B}_B$ and every $\epsilon > 0$,

$$\mu(A + \epsilon O_H) \leq \Phi(\Phi^{-1}(\mu(A)) + \epsilon)$$

Equivalance and Singularity

Theorem 2.6.13 (Cameron-Martin formula) Let B ba a separable Banach space, let μbe a centered Gaussian Borel meaure on B, Let H be its RKHS and let $h \in H$. Then the probability measure $\tau_h \mu$ defined as $\tau_h \mu(A) = \mu(A - h)$, is absolutely continuous with repect to μ, and

$$\frac{d\tau_h \mu}{d\mu}(x) = e^{(\phi^{-1} h(x) - \|h\|_H^2/2}$$

Moreover, if $\nu \notin H$, then $\tau_\nu \mu$ and μ are mutually singular.

Remark 2.6.14 $\tau_h \mu$ and μ are mutually absolutely continuous for any $h \in H$

Equivalance and Singularity

Corollary 2.6.17 Let μ be a centered Gaussian measure on a separable Banach space B, and let H be its RKHS. Then the support of μ is $\bar{H}$, the closure in B of H

The probability of small ball

Corollary 2.6.18 Let $C \subset B$ be a symmetric Borel set, where B is a separable Banach space, and let X be a centered Gaussian B-valued random variable. Then, for every $h \in H$,

$$\mathbb{P}(X - h \in C) \geq e^{-\|h\|_H^2/2}\mathbb{P}(X \in C)$$

The probability of small ball

Given a centered Gaussian B-valued random variable X with law μ, define its concentration function $\phi_x(\epsilon) = inf_{h\in H, \|h-x\|\leq\epsilon}[\frac{1}{2}\|h\|_H^2 - log\mathbb{P}(\|X\| < \epsilon)]$

Proposition 2.6.19 Let X be a centered Gaussian B-valued random variable, where B is a separable Banach space. Let $x \in supp(\mathcal{L}(X)) = \bar{H}$(See Cor 2.6.17) and $\epsilon > 0$. Then,

$$\phi_x(\epsilon) \leq -log\mathbb{P}(\|X - x\| < \epsilon) \leq \phi_x(\frac{\epsilon}{2})$$

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Review : Brownian motion

Brownian motion on $[0,1]$ is a centered sample continuous Gaussian process W whose covariance is $\mathbb{E}W_sW_t = s \wedge t$. It can be thought as a B-valued random variable where B=$C([0,1])$ endowed with sup norm

- The sample paths of W are all in $C^\alpha([0,1])$, the space of Hölder continuous of order α on all $0 < \alpha < 1/2$. (Exercise 2.3.2)

Released Process

- Let $(I_{0+}f)(t) = \int_0^t f(x)dx$ denote the primirive of f which is zero at zero for any continuous function f on [0,1], and let $(I_{0+}^k f)(t) = \int_0^t I_{0+}^{k-1} f(s)ds$
- $I_{0+}^0 W = W, (I_{0+}^k W)(t) = \int_0^t (I_{0+}^{k-1} W)(s)ds$
- $I_{0+}^k W$ are almost all in $C^{k+\alpha}([0,1])$
- **Released Process** Define $W^k(t) = \sum_{j=0} kt^j g_j/j! + (I_{0+}^k W)(t), t \in [0,1], k \geq 0$ where g_i are i.i.d standard normal variables idependent of W

RKHS of W^k

Proposition 2.6.24 For $k \geq 0$, the RKHS of W^k as a $C([0,1])$-valued random variable is

$$H_{W,k} = \{f : [0,1] \to \mathbb{R} : fisktimesdifferentiable, f^{(k)} isabs.cont.andf^{(k+1)} \in L^2([0,1])\}$$

with inner product $\langle f, g \rangle_{H_{W,k}} = \sum_{j=0} k f^{(j)}(0) g^{(j)}(0) + \int_0^1 f^{(k+1)}(s) g^{(k+1)}(s) ds$

Theorem 2.6.26 Let W be Brownian motion on [0,1] Then, there exists $C \in (0, \infty)$ such that, for all $0 < \epsilon \leq 1$,

$$-C\epsilon^{-2} \leq log\mathbb{P}\{sup_{t\in[0,1]}|W(t)| < \epsilon\} \leq -\frac{1}{C}\epsilon^{-2}$$

That is, the exact order of small ball concentration function ϕ_0^W of Brownian motion is $\phi_0^W = O(\epsilon^{-2})$ as $\epsilon \to 0$.

Theorem 2.6.29 If there is $\gamma > 0$ such that, for $C_1 < \infty$ and $\tau_i > 0$,

$$\phi_0(\epsilon) \le C_1 \epsilon^{-\gamma}, 0 < \epsilon \le \tau_1$$

and if

$$logN(H_1, \epsilon) \le C_2 \epsilon^{-\alpha}, 0 < \epsilon < \tau_2$$

for some $0 < \alpha < 2$, then there exists $C_3 < \infty$ such that for every $0 < \epsilon < \tau_3$,

$$\phi_0(\epsilon) \le C_3 \epsilon^{-2\alpha/(2-\alpha)}.$$